

Length and time scale divergences at the magnetization-reversal transition in the Ising model

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Abstract

The divergences of both the length and time scales, at the magnetization- reversal transition in Ising model under a pulsed field, have been studied in the linearized limit of the mean field theory. Both length and time scales are shown to diverge at the transition point and it has been checked that the nature of the time scale divergence agrees well with the result obtained from the numerical solution of the mean field equation of motion. Similar growths in length and time scales are also observed, as one approaches the transition point, using Monte Carlo simulations. However, these are not of the same nature as the mean field case. Nucleation theory provides a qualitative argument which explains the nature of the time scale growth. To study the nature of growth of the characteristic length scale, we have looked at the cluster size distribution of the reversed spin domains and defined a pseudo-correlation length which has been observed to grow at the phase boundary of the transition.

The dynamic response of pure Ising systems to time dependent magnetic fields is currently being studied intensively [1]. In particular, the response of Ising systems to pulsed fields has recently been investigated [2, 3, 4]. The pulse can be either “positive” or “negative”. At temperatures T below the critical temperature T_c of the corresponding static case (without any external field), the majority of the spins orient themselves along a particular direction giving rise to the prevalent order. In the following, we denote by positive (or negative) pulse an external field pulse applied along (or opposite) the direction of the existing order. The effects of a positive pulse can be analyzed by extending appropriately the finite size scaling technique to this finite time window case [2], and it does not involve any new transition or introduce any new thermodynamic scale into the problem. The negative field pulse, on the other hand, induces a new dynamic “magnetization-reversal” transition, involving completely new length and time scales [3, 4]. In fact, we believe, the spontaneously occurring dynamic symmetry-breaking transition in Ising models under (high frequency) external oscillating fields [1, 5] occurs actually during this “negative” pulse period (and not during the “positive” pulse period; compared to the instantaneous existing order in the system), and the universality classes of these two transitions are identical. We report here the results of an investigation on the nature of the characteristic length and time scales involved in this dynamic magnetization-reversal transition in an Ising model under the negative pulsed field.

In the absence of any symmetry breaking field, for temperatures below the critical temperature of the corresponding static case ($T < T_c$), there are two equivalent free energy minima with average magnetizations $+m_0$ and $-m_0$. If in the ordered state the equilibrium magnetization is $+m_0$ (say) and a very weak pulse is applied in the direction opposite to the existing order, then temporarily during the pulse period the free energy minimum with magnetization $-m_0$ will be brought down compared to that with $+m_0$. If this asymmetry is made permanent, then any non-vanishing field (strength), which is responsible for the asymmetry, would eventually induce a transition from $+m_0$ to $-m_0$ (in the limit of vanishing field strength). Instead, if the field is applied in the form of a pulse, the asymmetry in the free energy wells is removed after a finite period of time. In that case, the point of interest lies in the combination of the pulse height or strength (h_p) and its width or duration (Δt) that can give rise to the transition from $+m_0$ to $-m_0$. We call this a magnetization-reversal transition. A crucial point about the transition

is that it is not necessary that the system should attain its final equilibrium magnetization $-m_0$ during the presence of the pulse; the combination of h_p and Δt should be such that the final equilibrium state is attained at any subsequent time, even a long time after the pulse is withdrawn (see Fig. 1). The “phase boundary”, giving the minimal combination of h_p and Δt necessary for the transition, depends on the temperature. As $T \rightarrow T_c$, the magnetization reversal transition occurs at lower values of h_p and/or Δt and the transition disappears at $T \geq T_c$.

In the present paper we present an argument that this dynamic transition corresponds to infinite time and length scales, all along the phase boundary in the $h_p - \Delta t$ plane at any temperature at $T < T_c$. We show that the relaxation time τ and the correlation length ξ both diverge as one approaches the phase boundary. In the mean field case, we show (using equations of motion linearized in the magnetization) that

$$\tau \sim \ln\left(\frac{1}{m_w}\right) \quad \text{and} \quad \xi \sim \sqrt{\ln\left(\frac{1}{m_w}\right)}$$

where m_w is the “order-parameter” for the transition, given by the magnetization at the time of withdrawal of the pulse, starting from m_0 , the equilibrium magnetization at the temperature $T(< T_c)$ (see Fig. 1). It may be noted that $m_w(T, h_p, \Delta t) = 0$ at the phase boundary of the magnetization-reversal transition.

We also show that ξ and τ grow sharply as one approaches the phase boundary in the Monte Carlo case as well, although the nature of the growths are different from the mean field case. We also study the shapes and sizes of the reversed spin domains as one approaches the spin-reversal transition phase boundary in the Monte Carlo case. We compare the observed growth in the relaxation time in this case with that predicted by the nucleation theory.

The Ising model in the presence of an external magnetic field is described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{(ij)} J_{ij} S_i S_j - \sum_i h_i S_i, \quad (1)$$

where S_i denotes the spin at i th site, J_{ij} is the cooperative interaction between the spins at sites i and j and $(...)$ denotes the nearest-neighbour pairs.

Here h_i is the external field, allowed to be time dependent, and also site-dependent to allow investigation of separation-dependent correlations. The free energy of the system in the Bragg-Williams approximation is given by[6]

$$F = -\frac{1}{2} \sum_{(ij)} J_{ij} m_i m_j - \sum_i h_i m_i + \sum_i \frac{T}{2} [\ln(1 - m_i^2) + m_i \ln \left(\frac{1 + m_i}{1 - m_i} \right) - 2 \ln 2], \quad (2)$$

with $m_i = \langle S_i \rangle$, where $\langle \dots \rangle$ denotes the thermal average. In the presence of a time and site-dependent field, the time dependent magnetization satisfies the Langevin equation

$$\frac{dm_i}{dt} = -\frac{\lambda}{T} \frac{\delta F}{\delta m_i} = \lambda \left[\sum_j K_{ij} m_j(t) + \frac{h_i(T)}{T} - \frac{1}{2} \ln \left(\frac{1 + m_i(t)}{1 - m_i(t)} \right) \right], \quad (3)$$

where $K_{ij} = J_{ij}/T$ and λ is a constant. Differentiation with the space and time dependent magnetic field $h_i(t)$ generates the space and time dependent susceptibility. After the differentiation we can set $h_i(t) = h(t)$ to obtain results for a pulsed field uniform in space. Then $m_i(t) \rightarrow m(t)$ gives

$$\frac{dm(t)}{dt} = \lambda \left[K(0)m(t) + \frac{h(t)}{T} - \frac{1}{2} \ln \left(\frac{1 + m(t)}{1 - m(t)} \right) \right]. \quad (4)$$

The resulting equation for the susceptibility, in the Fourier space, is

$$\frac{d\chi_q(t)}{dt} = \lambda \left[K(q) - \frac{1}{1 - m^2(t)} \right] \chi_q(t) + \frac{\lambda}{T} \delta(t - t'). \quad (5)$$

Here, $K(q)$ is the Fourier transform of K_{ij} ; for small q , $K(q) \simeq K(0)(1 - q^2)$; in the mean field theory $K(0) = T_c/T$. Using (4) and (5), we can write

$$\frac{d\chi_q(t)}{dm(t)} = \frac{\left[K(q) - \frac{1}{1 - m^2(t)} \right] \chi_q(t)}{K(0)m(t) - \frac{h_p}{T} - \frac{1}{2} \ln \left[\frac{1 + m(t)}{1 - m(t)} \right]}. \quad (6)$$

In the limit when $m(t)$ is small, retaining up to the linear term in $m(t)$,

$$\frac{d\chi_q(t)}{dm(t)} = \frac{[K(q) - 1]\chi_q(t)}{[K(0) - 1]m(t) - \frac{h_p}{T}}. \quad (7)$$

This equation can now be solved in the three different time zones (Fig. 1): namely, in the equilibrium regime before the application of the pulse where $m = m_0$ (regime I), the (nonequilibrium) pulsed period regime, at the end of which $m = m_w$ (regime II), and the regime after the pulse is withdrawn (regime III) when the system eventually returns to equilibrium (with $m(t \rightarrow \infty) = -m_0$ if the transition occurs, or $= m_0$ if it does not). Hence in regime II and III, we get the non-equilibrium susceptibility χ_q as a function of $m(t)$. The solution of (4) also gives the non-equilibrium magnetization $m(t)$, and hence we can also arrive at $\chi_q(t)$. Noting that $\chi_q(t) = \chi_q^s$ when $m(t) = m_0$, at the start of regime II, where m_0 and χ_q^s are equilibrium values of the magnetization and susceptibility respectively, we can integrate (7) in that regime to obtain

$$\frac{\chi_q(t)}{\chi_q^s} = \left[\frac{m(t) - \Gamma}{m_0 - \Gamma} \right]^{a_q}, \quad (8)$$

where

$$\Gamma = \frac{h_p/T}{K(0) - 1} \quad (8a)$$

and

$$a_q = \frac{K(q) - 1}{K(0) - 1}. \quad (8b)$$

Also integrating the linearized version of (4) in region II, one gets

$$m(t) = \Gamma + (m_0 - \Gamma) \exp[\lambda b(t - t_0)], \quad (9)$$

where $b = K(0) - 1$. At the end of region II, the value of magnetization is given by

$$m_w = m(t_0 + \Delta t) = \Gamma + (m_0 - \Gamma) \exp(\lambda b \Delta t). \quad (10)$$

The eqn. (8) can therefore be written as

$$\frac{\chi_q(t)}{\chi_q^s} = \exp(\lambda b a_q t) = \exp[(K(q) - 1)\lambda t]. \quad (11)$$

In regime III, however, $h(t) = 0$ and the (initial) boundary condition is $m(t_0 + \Delta t) = m_w$. Integrating (7) in this regime one gets

$$\frac{\chi_q(t)}{\chi_q(t_0 + \Delta t)} = \left[\frac{m(t)}{m_w} \right]^{a_q}$$

or

$$\chi_q(t) = \chi_q^s \exp[\lambda(K(q) - 1)(t + \Delta t)] \left[\frac{m(t)}{m_w} \right]^{a_q}, \quad (12)$$

where use has been made of the eqn. (8). Concentrating on the dominating q -dependence of the susceptibility, one can write

$$\chi_q(t) \sim \chi_q^s \exp[-q^2 \xi^2], \quad (13)$$

where the correlation length ξ is defined as

$$\xi \equiv \xi(m_w) = \left[\frac{\ln(1/m_w)}{1 - T/T_c} \right]^{\frac{1}{2}}. \quad (14)$$

This is one of the principal results of this paper, and it shows that the characteristic length ξ diverges as the order parameter m_w goes to zero.

Consider now the t dependence arising in $\chi_{q=0}(t)$ through the factor $m(t)^{a_q}$. Solving (4) in regime III yields

$$m(t) = m_w \exp[\lambda b\{t - (t_0 + \Delta t)\}], \quad (15)$$

which shows that long time is required to attain moderate values of $m(t)$ starting from low values of m_w . Especially, starting from time $t = t_0 + \Delta t$, the time taken by the system to reach the final equilibrium value is defined as the relaxation time τ of the system. Therefore from (15) we can write

$$\tau = \frac{1}{\lambda} \left(\frac{T}{T_c - T} \right) \ln \left(\frac{m_0}{m_w} \right). \quad (16)$$

The growth of the time scale occurs in $\chi_{q=0}(t)$ too through the $m(t)$ dependence :

$$\chi_{q=0}(t) \sim \left[\frac{m(t)}{m_w} \right]^{a_{q=0}} \sim \exp[\lambda b\{t - (t_0 + \Delta t)\}].$$

Eqn. (14) and (16) can be used to establish a relationship between τ and ξ :

$$\tau \sim \ln \left(\frac{1}{m_w} \right) \sim \frac{T}{T_c} \xi^2. \quad (17)$$

This corresponds to critical slowing, with the characteristic time diverging with the characteristic length with the dynamical critical exponent

$$z = 2.$$

The above results are obtained in the linearized limit of the mean field eqns. of motion (4) and (5). We also measured, solving the full dynamical equation (4) numerically, the relaxation time τ by computing the time required by $m(t)$ to reach the final equilibrium value $\pm m_0$, with an accuracy of $O(10^{-4})$, from the time of withdrawal of the pulse (in regime III). Fig. 2 shows that this τ indeed diverges as one approaches the phase boundary, where $m_w = 0$. In fact, the numerical results are observed to fit very well with the analytic result (17) (shown by the solid line in Fig. 2).

The divergence of both the time and length scale were also investigated at low temperatures by employing Monte Carlo methods. Simulations [4] on a square lattice of typical size $L = 200$ with periodic boundary conditions indicated an exponential growth of the time scale :

$$\tau \sim \exp[-c(T) | m_w |], \quad (18)$$

where $c(T)$ is a constant depending on temperature only. Further, finite size scaling of the order parameter relationship

$$m_w \sim | h_p - h_p^c |^\beta \quad (19)$$

is consistent[4] with $\beta = 0.90 \pm 0.02$ and with a correlation length divergence with $\nu = 1.5 \pm 0.3$. (Here h_p^c is the critical value of the pulse field h_p , making $m_w = 0$ at the end of regime II). These results qualitatively compare with the divergence of scales at the transition point predicted by the mean field treatment. However, the growths of the time and length scales are quantitatively of different nature to that of the mean field case, because at low temperatures droplet growth is a dominant mechanism. The growth of droplets of size l is associated with an activation energy[4] $E(l) = -2h_p l^d + \sigma l^{d-1}$, where σ is the surface tension. Using the relationship between l and h_p at the energy minimum together with (18) at small m_w gives a characteristic time

$$\tau \sim \exp \left[\frac{1}{T} h_p^{1-d} \right] \sim \exp[-c_1(T) | m_w |^{1/\beta} (h_p^c)^{d-2}]. \quad (20)$$

Since β is close to unity, this is consistent with the observed relation (18).

The typical size of cluster or domain of reversed spins provides a qualitative idea about the correlation length of the system. In order to study the growth of the typical reversed-spin domain size, we define a pseudo-correlation length $\tilde{\xi}$ as follows :

$$\tilde{\xi}^2 = \frac{\sum_s R_s^2 s^2 n_s}{\sum_s s^2 n_s}, \quad (21)$$

where n_s is the number of domains or clusters of size s and the radius of gyration R_s is defined as $R_s^2 = \sum_{i=1}^s |r_i - r_0|^2 / s$, where r_i is the position vector of the i th spin of the cluster and $r_0 = \sum_{i=1}^s (r_i / s)$ is defined as the centre of mass of the particular cluster. The pseudo-correlation length $\tilde{\xi}$ is observed to grow to system size order as one approaches the phase boundary (Fig. 3); thereby providing further indication of the growth of a length scale. It should be noted that, as in the static transition in the pure Ising system, the length $\tilde{\xi}$ is distinct from the correlation length [7].

In the linear limit of the mean field dynamics, it has been possible to show the divergence of both the length and time scales at the magnetization-reversal transition phase boundary. Sharp growth of these scales has also been observed in the Monte Carlo case, studied in two dimension. Here, we looked at the size distribution of the clusters or domains of reversed spins whose average size was observed to grow at the phase boundary of the transition.

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Figure Captions

FIG. 1. Schematic time variation of the pulsed field $h(t)$ and the corresponding response magnetization $m(t)$ for two different cases. The solid line indicates no magnetization-reversal case whereas the dashed line indicates a magnetization-reversal.

FIG. 2. Divergence of the relaxation time in mean field limit for $T = 0.8$ and $\Delta t = 20$ (from numerical solution of eqn. (4)). The solid line indicates the corresponding analytical estimate (eqn. (15)).

FIG. 3. Growth of the pseudo-correlation length $\tilde{\xi}$ for different system sizes in the Monte Carlo study on a square lattice of size $L \times L$.

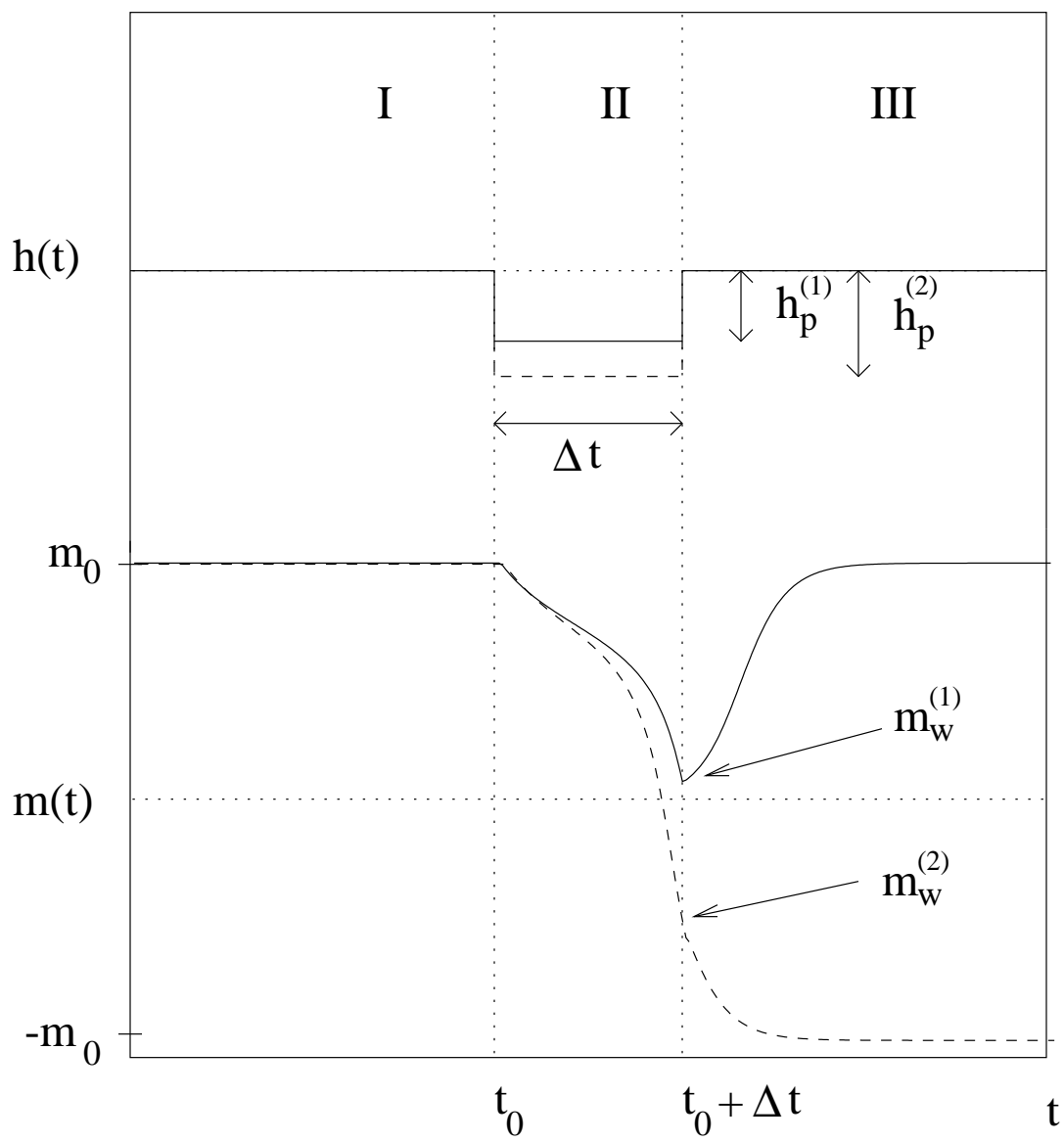


FIG. 1

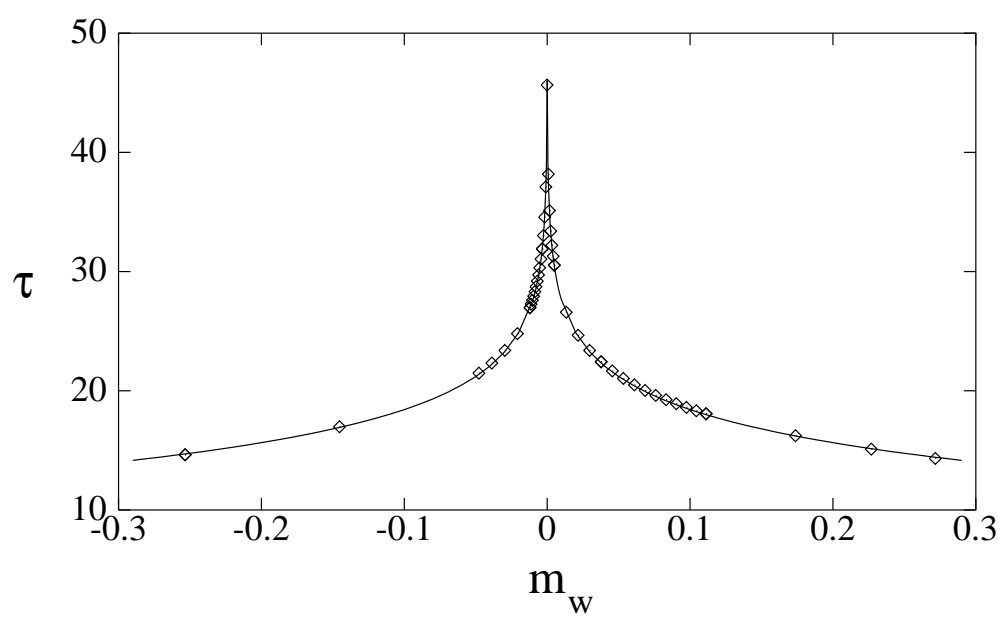


FIG. 2

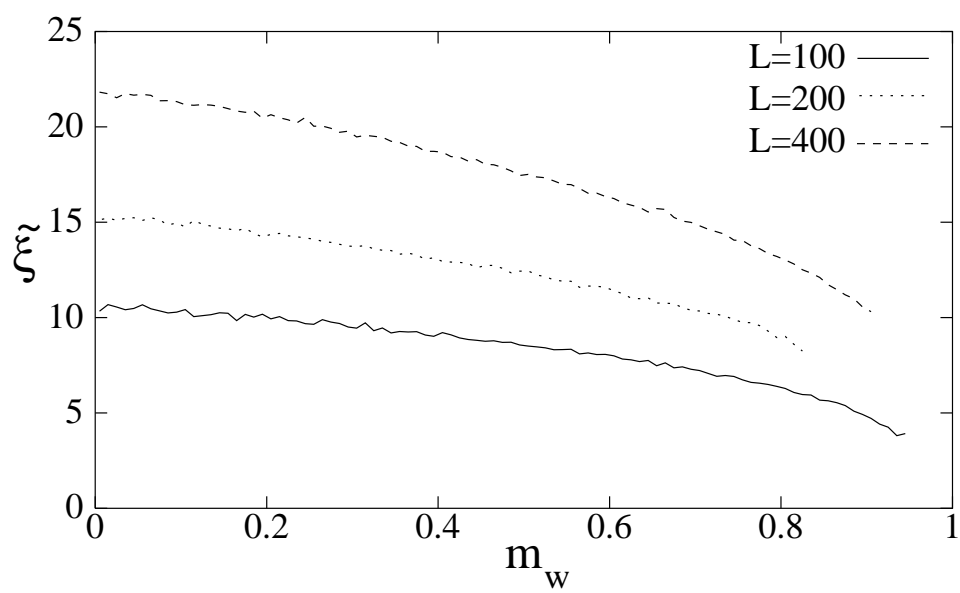


FIG. 3